B3 - Games, Graphs and Algebra : Back-paper Exam

Yogeshwaran D.

December 30, 2024. Time : 10.00 - 12.30 PM. **Maximum points : 50**

2 points will be deducted if you do not write your name on the answerscript.

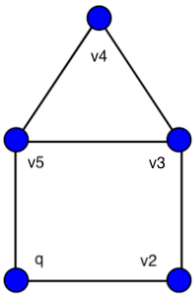
ALL QUESTIONS CARRY 10 POINTS. ATTEMPT ALL OF THEM.

You are free to use any results that you have learnt in your course but please cite them clearly. Provide as many details as you can. Some simple notions and notations are recalled at the end.

1. (a) Let C_n denote the cycle graph with vertices $v_0, \ldots, v_{n-1}$ arranged counter-clockwise in a circle. Identify each divisor D with a vector $(D(v_0), \ldots, D(v_{n-1}))$. Show that the two divisors D and D' on C_n of the same degree are linearly equivalent iff $D \cdot (0, \ldots, n-1) = D' \cdot (0, \ldots, n-1) \bmod n$. **(6)**

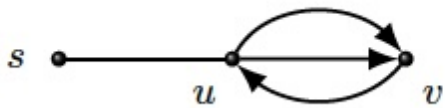
   (b) If D is a unwinnable divisor, show that there exists a maximal unwinnable divisor N such that $N \geq D$. **(4)**

2. Consider the house graph as below. Find all effective divisors of degree 2 and rank 1.



3. Let $\tilde{V} = [m] \times [n]$ be the grid with usual nearest-neighbour edges. All the vertices on the boundary of $\tilde{V}$ (i.e., $(i,j) \in \tilde{V}$ such that either $i \in \{0, m\}$ or $j \in \{0, n\}$) are connected to a sink s. Show that $\overrightarrow{1}$ is not recurrent and $\overrightarrow{2}$ is recurrent.

4. A stable sandpile is recurrent iff it has no Forbidden subconfiguration. Prove or disprove this claim for the following graphs - (i) G be a directed acyclic sandpile graph with a selfish vertex and (ii) Extended path graph as drawn below.



5. For $n \geq 1$, consider the cycle graph C_n as a directed graph by orienting each edge in both directions. Compute threshold density $\zeta_\tau(D)$ for divisors $D = \overrightarrow{1}$, the all 1 sandpile and $D = \overrightarrow{0}$, the zero sandpile.

NOTIONS AND NOTATION

- All graphs G are finite, connected multi-graphs (undirected) or multi-digraphs (i.e., directed multi-graphs) depending on context. deg denotes degree of a vertex in the graph i.e., the number of edges incident on the graph.
- $r(D)$ denotes rank of a divisor $D = \sum_v D(v)v$. $\deg(D) = \sum_v D(v)$ denotes degree os a divisor D.
- A sandpile graph G is (V, E, s) where (V, E) form a multi-digraph with a globally accessible sink s.

- A multidigraph is acyclic if there are no directed cycles.
- A stable sandpile c is recurrent if for all $a \geq 0$, there exists $b \geq 0$ such that $(a+b)^\circ = c$.
- A fixed energy sandpile is a sandpile on a multi-digraph $G = (V, E)$ but without sinks.
- The threshold density with respect to divisor D is $\zeta_\tau(D) := \mathbb{E}_D \frac{\deg(D_\tau)}{|V|}$, where D_τ is the first alive divisor in the fixed energy sandpile Markov chain when starting with divisor D.